Section A: Pure Mathematics

1 Let

$$\mathrm{f}(x) = \sin^2 x + 2\cos x + 1$$

for $0 \leqslant x \leqslant 2\pi$. Sketch the curve $y = \mathrm{f}(x)$, giving the coordinates of the stationary points. Now let

$$\mathrm{g}(x) = \frac{a\mathrm{f}(x) + b}{c\mathrm{f}(x) + d} \qquad ad \neq bc\,,\ d \neq -3c\,,\ d \neq c\,.$$

Show that the stationary points of $y = \mathrm{g}(x)$ occur at the same values of x as those of $y = \mathrm{f}(x)$, and find the corresponding values of $\mathrm{g}(x)$.

Explain why, if $d/c < -3$ or $d/c > 1$, $|\mathrm{g}(x)|$ cannot be arbitrarily large.

2 Let

$$\mathrm{I}(a, b) = \int_0^1 t^a (1-t)^b \,\mathrm{d}t \qquad (a \geqslant 0,\ b \geqslant 0).$$

(i) Show that $\mathrm{I}(a, b) = \mathrm{I}(b, a)$,

(ii) Show that $\mathrm{I}(a, b) = \mathrm{I}(a+1, b) + \mathrm{I}(a, b+1)$.

(iii) Show that $(a+1)\mathrm{I}(a, b) = b\mathrm{I}(a+1, b-1)$ when a and b are positive and hence calculate $\mathrm{I}(a, b)$ when a and b are positive integers.

3 The value V_N of a bond after N days is determined by the equation

$$V_{N+1} = (1+c)V_N - d \qquad (c > 0,\ d > 0),$$

where c and d are given constants. By looking for solutions of the form $V_T = Ak^T + B$ for some constants A, B and k, or otherwise, find V_N in terms of V_0.

What is the solution for $c = 0$? Show that this is the limit (for fixed N) as $c \to 0$ of your solution for $c > 0$.

4 Show that the equation (in plane polar coordinates) $r = \cos\theta$, for $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$, represents a circle.

Sketch the curve $r = \cos 2\theta$ for $0 \leqslant \theta \leqslant 2\pi$, and describe the curves $r = \cos 2n\theta$, where n is an integer. Show that the area enclosed by such a curve is independent of n.

Sketch also the curve $r = \cos 3\theta$ for $0 \leqslant \theta \leqslant 2\pi$.

5 The exponential of a square matrix $\mathbf{A}$ is defined to be

$$\exp(\mathbf{A}) = \sum_{r=0}^{\infty} \frac{1}{r!}\mathbf{A}^r ,$$

where $\mathbf{A}^0 = \mathbf{I}$ and $\mathbf{I}$ is the identity matrix.

Let

$$\mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

Show that $\mathbf{M}^2 = -\mathbf{I}$ and hence express $\exp(\theta\mathbf{M})$ as a single 2×2 matrix, where θ is a real number. Explain the geometrical significance of $\exp(\theta\mathbf{M})$.

Let

$$\mathbf{N} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

Express similarly $\exp(s\mathbf{N})$, where s is a real number, and explain the geometrical significance of $\exp(s\mathbf{N})$.

For which values of θ does

$$\exp(s\mathbf{N})\ \exp(\theta\mathbf{M})\ =\ \exp(\theta\mathbf{M})\ \exp(s\mathbf{N})$$

for all s? Interpret this fact geometrically.

6 (i) Show that four vertices of a cube, no two of which are adjacent, form the vertices of a regular tetrahedron. Hence, or otherwise, find the volume of a regular tetrahedron whose edges are of unit length.

(ii) Find the volume of a regular octahedron whose edges are of unit length.

(iii) Show that the centres of the faces of a cube form the vertices of a regular octahedron. Show that its volume is half that of the tetrahedron whose vertices are the vertices of the cube.

[*A regular tetrahedron (octahedron) has four (eight) faces, all equilateral triangles.*]

7 Sketch the graph of $\mathrm{f}(s) = e^s(s-3)+3$ for $0 \leqslant s < \infty$. Taking $e \approx 2.7$, find the smallest positive integer, m, such that $\mathrm{f}(m) > 0$.

Now let

$$\mathrm{b}(x) = \frac{x^3}{e^{x/T}-1}$$

where T is a positive constant. Show that $\mathrm{b}(x)$ has a single turning point in $0 < x < \infty$. By considering the behaviour for small x and for large x, sketch $\mathrm{b}(x)$ for $0 \leqslant x < \infty$.

Let

$$\int_0^\infty \mathrm{b}(x)\,\mathrm{d}x = B,$$

which may be assumed to be finite. Show that $B = KT^n$ where K is a constant, and n is an integer which you should determine.

Given that $B \approx 2\int_0^{Tm} \mathrm{b}(x)\mathrm{d}x$, use your graph of $\mathrm{b}(x)$ to find a rough estimate for K.

8 (i) Show that the line $\mathbf{r} = \mathbf{b} + \lambda\mathbf{m}$, where $\mathbf{m}$ is a unit vector, intersects the sphere $\mathbf{r}.\mathbf{r} = a^2$ at two points if

$$a^2 > \mathbf{b}.\mathbf{b} - (\mathbf{b}.\mathbf{m})^2\,.$$

Write down the corresponding condition for there to be precisely one point of intersection. If this point has position vector $\mathbf{p}$, show that $\mathbf{m}.\mathbf{p} = 0$.

(ii) Now consider a second sphere of radius a and a plane perpendicular to a unit vector $\mathbf{n}$. The centre of the sphere has position vector $\mathbf{d}$ and the minimum distance from the origin to the plane is l. What is the condition for the plane to be tangential to this second sphere?

(iii) Show that the first and second spheres intersect at right angles (*i.e.* the two radii to each point of intersection are perpendicular) if

$$\mathbf{d}.\mathbf{d} = 2a^2\,.$$

Section B: Mechanics

9 A uniform right circular cone of mass m has base of radius a and perpendicular height h from base to apex. Show that its moment of inertia about its axis is $\frac{3}{10}ma^2$, and calculate its moment of inertia about an axis through its apex parallel to its base.
[*Any theorems used should be stated clearly.*]

The cone is now suspended from its apex and allowed to perform small oscillations. Show that their period is

$$2\pi\sqrt{\frac{4h^2+a^2}{5gh}}\,.$$

[*You may assume that the centre of mass of the cone is a distance* $\frac{3}{4}h$ *from its apex.*]

10 Two identical spherical balls, moving on a horizontal, smooth table, collide in such a way that both momentum and kinetic energy are conserved. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be the velocities of the balls before the collision and let $\mathbf{v}_1'$ and $\mathbf{v}_2'$ be the velocities of the balls after the collision, where $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_1'$ and $\mathbf{v}_2'$ are two-dimensional vectors. Write down the equations for conservation of momentum and kinetic energy in terms of these vectors. Hence show that their relative speed is also conserved.

Show that, if one ball is initially at rest but after the collision both balls are moving, their final velocities are perpendicular.

Now suppose that one ball is initially at rest, and the second is moving with speed V. After a collision in which they lose a proportion k of their original kinetic energy ($0 \leqslant k \leqslant 1$), the direction of motion of the second ball has changed by an angle θ. Find a quadratic equation satisfied by the final speed of the second ball, with coefficients depending on k, V and θ. Hence show that $k \leqslant \frac{1}{2}$.

11 Consider a simple pendulum of length l and angular displacement θ, which is **not** assumed to be small. Show that

$$\frac{1}{2}l\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = g(\cos\theta - \cos\gamma)\,,$$

where γ is the maximum value of θ. Show also that the period P is given by

$$P = 2\sqrt{\frac{l}{g}}\int_0^\gamma \big(\sin^2(\gamma/2) - \sin^2(\theta/2)\big)^{-\frac{1}{2}}\,\mathrm{d}\theta\,.$$

By using the substitution $\sin(\theta/2) = \sin(\gamma/2)\sin\phi$, and then finding an approximate expression for the integrand using the binomial expansion, show that for small values of γ the period is approximately

$$2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\gamma^2}{16}\right)\,.$$

Section C: Probability and statistics

12 The mountain villages A, B, C and D lie at the vertices of a tetrahedron, and each pair of villages is joined by a road. After a snowfall the probability that any road is blocked is p, and is independent of the conditions of any other road. The probability that, after a snowfall, it is possible to travel from any village to any other village by some route is P. Show that

$$P = 1 - p^2(6p^3 - 12p^2 + 3p + 4).$$

13 Write down the probability of obtaining k heads in n tosses of a fair coin. Now suppose that k is known but n is unknown. A *maximum likelihood estimator* (MLE) of n is defined to be a value (which must be an integer) of n which maximizes the probability of k heads.

A friend has thrown a fair coin a number of times. She tells you that she has observed one head. Show that in this case there are *two* MLEs of the number of tosses she has made.

She now tells you that in a repeat of the exercise she has observed k heads. Find the two MLEs of the number of tosses she has made.

She next uses a coin biased with probability p (known) of showing a head, and again tells you that she has observed k heads. Find the MLEs of the number of tosses made. What is the condition for the MLE to be unique?

14 A hostile naval power possesses a large, unknown number N of submarines. Interception of radio signals yields a small number n of their identification numbers X_i $(i = 1, 2, ..., n)$, which are taken to be independent and uniformly distributed over the continuous range from 0 to N. Show that Z_1 and Z_2, defined by

$$Z_1 = \frac{n+1}{n}\text{Max}\{X_1, X_2, ..., X_n\} \qquad \text{and} \qquad Z_2 = \frac{2}{n}\sum_{i=1}^{n} X_i \ ,$$

both have means equal to N.

Calculate the variance of Z_1 and of Z_2. Which estimator do you prefer, and why?